MATH 504 HOMEWORK 8

Due Monday, November 26.

Problem 1. *Suppose that for all* n, $2^{\aleph_n} = \aleph_{\omega+1}$. *Show that* $2^{\aleph_\omega} = \aleph_{\omega+1}$.

Hint: for every $A \subset \aleph_\omega$, define $A_n = A \cap \aleph_n$. Consider the map $h(A) = \langle A_n \mid n < \omega \rangle$.

Problem 2. *Suppose that* $i : \mathbb{P} \to \mathbb{Q}$ *is such that:*

(1) $i(1_\mathbb{P}) = 1_\mathbb{Q}$
(2) *for all* $p_1, p_2 \in \mathbb{P}$, *if* $p_1 \leq p_2$, *then* $i(p_1) \leq i(p_2)$,
(3) *for all* $p_1, p_2 \in \mathbb{P}$, $p_1 \perp p_2$ *iff* $i(p_1) \perp i(p_2)$,
(4) *whenever* A *is a maximal antichain in* $\mathbb{P}$, *then* $i"A = \{i(p) \mid p \in A\}$ *is a maximal antichain.*

Prove that if G *is* $\mathbb{Q}$*-generic, then .* $H =_{def} \{p \in \mathbb{P} \mid i(p) \in G\}$ *is* $\mathbb{P}$*-generic. Also show that* $V[H] \subset V[G]$. *Here* V *is some ground model.*

Problem 3. *Let* $\kappa < \lambda$ *be such that* κ *is a regular cardinal,* λ *is an inaccessible cardinal, and let* $\kappa < \tau < \lambda$. *Show that there is* $i : Col(\kappa, \tau) \to Col(\kappa, < \lambda)$ *as in the above problem.*

Problem 4. *Suppose that* κ *is inaccessible in* M *and* $\mathbb{P}$ *is a poset of cardinality less than* κ. *Show that for any generic* G *for* $\mathbb{P}$ *over* M, *in* $M[G]$, κ *remains inaccessible.*

Problem 5. *Suppose that in* M, κ *is a regular cardinal,* $\mathbb{P}$ *has the* κ*-c.c. and* G *is* $\mathbb{P}$ *over* M.

(1) *Suppose that in* $M[G]$, C *is a club subset of* κ. *Show that* $D := \{\alpha < \kappa \mid 1_\mathbb{P} \Vdash \alpha \in \dot{C}\}$ *is a club subset of* κ *in* M.
(2) *Use the above to show that* $\mathbb{P}$ *preserves stationary sets i.e. if* $M \models$ *"*$S \subset \kappa$ *is stationary", then* $M[G] \models$ *"*$S \subset \kappa$ *is stationary".*